

# **A Path-Based Algorithm for the Cross Nested Logit Stochastic User Equilibrium Traffic**

## **Assignment**

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### **ABSTRACT**

This paper investigates the single-class static stochastic user equilibrium (SUE) problem with separable and additive link costs. A SUE assignment based on the Cross-Nested Logit (CNL) route choice model is presented. The CNL model can better represent route choice behavior compared to the Multinomial Logit (MNL) model, while keeping a closed form equation. The paper uses a specific optimization formulation developed for the CNL model, and develops a path-based algorithm for the solution of the CNL-SUE problem based on adaptation of the disaggregate simplicial decomposition (DSD) method. The paper illustrates the algorithmic implementation on a real size network and discusses the trade-offs between MNL-SUE and CNL-SUE assignment.

## **1 INTRODUCTION**

Static traffic assignment is the problem of loading demand for travel on a transportation network. Traffic assignment heavily depends on the route choice model being used to allocate origin-to-destination (OD) flows to the various routes in the network. For example, the deterministic traffic assignment model assumes that drivers have complete and accurate information on the state of the network when they make their route choices. This implies that they consider the universal set of all available routes in the network, and are able to exactly select optimal routes among them. This model, as well as most assignment procedures, also assumes the following regarding the drivers and the attributes of the available routes: (i) all drivers apply similar choice mechanisms, (ii) route travel costs are the sum of the travel costs on the links that comprise the route, and (iii) link travel costs only depend on properties of the link itself. Relaxation of assumption (i) leads to multi-class models, while relaxation of (ii) or (iii) leads to non-separable assignment models.

Stochastic user equilibrium (SUE) models relax the assumption of optimal route choices. Instead, probabilistic choice models are applied to predict the share of drivers that use the various routes in the choice set. The model structures and explanatory variables that capture route choices have received significant attention in the behavioral literature. However, the impact of the assumptions made in constructing these models and in determining the choice

sets they are used with on traffic assignment results have not been studied empirically on real-world networks. This issue is of considerable practical importance since the solution of these more sophisticated and realistic user equilibrium problems may be significantly more expensive computationally.

Most of the applications of SUE models reported in the literature are based on the multinomial logit model (MNL). Special properties of the MNL model enable the implementation of efficient link-based solution algorithms that avoid explicit enumeration of the route choice set to solve the MNL-SUE problem. However, these algorithms assume an implicit choice set, such as the use of all efficient paths (Maher 1998, Dial 2001), or all cyclic and acyclic paths (Bell 1995, Akamatsu 1996). These choice sets are unrealistic from a behavioral standpoint. Path-based algorithms allow a more flexible definition of the choice set. Furthermore, The MNL model exhibits the property of independence of irrelevant alternatives (IIA), which is undesired in the context of route choices since it does not account for similarities among routes. Daganzo and Sheffi (1977) demonstrate this deficiency of the MNL model for route choice using simple network examples. Instead, they proposed to use the multinomial probit model (MNP). However, the MNP model is computationally unattractive since choice probabilities cannot be expressed in closed form. In recent years, a number of other discrete choice model structures were adapted to route choice behavior. Modifications to

MNL, such as C-logit (Cascetta and al., 1996) and path-size logit (Ben-Akiva and Bierlaire, 1999, Ramming, 2001) account for similarities among routes through additional terms in the systematic utilities of the various routes. More general correlation structures may be captured using generalized extreme value (GEV) models, such as paired combinatorial logit (Prashker and Bekhor, 1998, Gliebe et al., 1999) and cross-nested logit (Vovsha and Bekhor, 1998, Prashker and Bekhor, 1998), and mixed logit (also termed logit kernel) models (Bekhor et al., 2002).

The mixed logit formulation is very flexible, however the model has no closed form and therefore simulation methods are needed to compute choice probabilities. Stochastic loading procedures that rely on this model are therefore computationally expensive. The CNL model combines a flexible error structure that can capture similarities among routes with a closed form equation. It is therefore further explored in the equilibrium context in this paper.

While formulations of the SUE assignment problem that correspond to some of these route choice models have been introduced in the literature (Bekhor and Prashker, 1999), there has been little study of the practical implications of using these models on the assignment results and even less development of appropriate algorithms for their solution.

This paper investigates the CNL-SUE traffic assignment model. We present an algorithm for the solution of this problem, investigate the impact of various parameters of the problem and

the definition of the path choice set on computational performance and compare the results to those obtained with MNL-SUE assignment. The paper is organized as follows. First, the CNL-SUE optimization formulation is presented and an algorithm developed to solve the CNL-SUE problem is described. The Winnipeg network, which is used as a case study is described next. The analysis that uses this network investigates three main issues related to model implementation: computational performance, comparison of the assignment results against MNL-SUE and the influence of the path set on the solution.

## 2 CNL-SUE ASSIGNMENT PROBLEM

The CNL model was adapted to the route choice problem by Prashker and Bekhor (1998) and Vovsha and Bekhor (1998). Their adaptation uses a two-level nesting structure in which the upper level (nests) includes all the links in the network. The lower level consists of all the routes in the set  $C$  of routes that connect between the origin and destination. Each route is allocated to all the nests (links) that it is composed of. Assuming this structure, the probability of choosing route  $k$  is given by:

$$P(k) = \frac{\exp \left[ -\theta c_k + \ln \sum_{m \in M} \alpha_{mk} \left( \sum_l \alpha_{ml} \exp(-\theta c_l) \right)^{\mu-1} \right]}{\sum_{j \in C} \exp \left[ -\theta c_j + \ln \sum_{m \in M} \alpha_{mj} \left( \sum_l \alpha_{ml} \exp(-\theta c_l) \right)^{\mu-1} \right]} \quad (1)$$

$c_k$  is the generalized cost of travel on path  $k$ .  $\theta$ ,  $\mu$ , and  $\alpha_{mk}$  are parameters of the model.  $\theta$  is a dispersion parameter that determines the sensitivity of route choice fractions to changes in travel costs. The parameter  $\mu$  indicates the degree of nesting, as in the nested logit model: when  $\mu=1$ , the model collapses to MNL, and when  $\mu$  tends to zero, the model becomes probabilistic at the higher (link) level and deterministic at the lower (path) level. This parameter can be nest-specific, as indicated by Bekhor and Prashker (2001), but for ease of interpretation we consider a single nesting coefficient. The parameters  $\alpha_{mk}$  determine the allocation of route  $k$  among the links  $m$  is composed of. Prashker and Bekhor (1998) proposed determination of these parameters exclusively based on network topology, which uses the physical length of the links that are common to various routes.

Bekhor and Prashker (1999) presented a mathematical program formulation, whose solution corresponds to the CNL-SUE assignment:

$$\begin{aligned}
Min \ Z &= Z_1 + Z_2 + Z_3 \\
Z_1 &= \sum_a \int_0^{x_a} c_a(w) dw \\
Z_2 &= \frac{\mu}{\theta} \sum_{rs} \sum_m \sum_k f_{mk}^{rs} \ln \frac{f_{mk}^{rs}}{(\alpha_{mk}^{rs})^{1/\mu}} \\
Z_3 &= \frac{1-\mu}{\theta} \sum_{rs} \sum_m \left( \sum_k f_{mk}^{rs} \right) \ln \left( \sum_k f_{mk}^{rs} \right) \\
s.t. \quad &\sum_m \sum_k f_{mk}^{rs} = q^{rs}, \quad \forall r, s \\
&f_{mk}^{rs} \geq 0, \quad \forall m, k, r, s
\end{aligned} \tag{2}$$

$x_a$  and  $c_a$  are the flow and cost on link  $a$ , respectively.  $f_{mk}^{rs}$  is a part of the flow on path  $k$  that is allocated to nest  $m$  between origin  $r$  and destination  $s$ .  $q^{rs}$  is the demand for travel from  $r$  to  $s$ . The expression  $f_{mk}^{rs} \ln \frac{f_{mk}^{rs}}{(\alpha_{mk}^{rs})^{1/\mu}}$  is defined as zero if either  $f_{mk}^{rs} = 0$  or  $\alpha_{mk}^{rs} = 0$ .

The objective function in the above formulation is composed of three terms. The first term ( $Z_1$ ) is identical to the deterministic user equilibrium formulation. The second term ( $Z_2$ ) is an entropy term similar to Fisk's formulation (Fisk 1980) for the MNL-SUE problem, but modified to include the allocation coefficients and the nesting coefficient. The third term ( $Z_3$ ) is also an entropy term, in which the flows  $f_{mk}^{rs}$  are aggregated by all routes. The constraints of the problem are conservation equations and non-negativity of path flows, which are similar to those in other mathematical formulations for the traffic assignment problem.

Assuming that the link costs are continuous monotonically increasing functions of link flows, Bekhor and Prashker (1999) show that the objective function is continuous and convex. In addition, if the nesting coefficient is equal to 1, the above formulation collapses to the MNL-SUE formulation.

Few studies have actually implemented the CNL-SUE. Bekhor and Prashker (1999) applied their formulation using the method of successive averages (MSA) algorithm of Sheffi and Powell (1982) to a small network. Chen et al. (2003) developed an algorithm based on the partial linearization method for solving the PCL-SUE problem, which is a special case of



equation (2). In this paper, we adapt the DSD algorithm (Damberg et al. 1996) to the CNL-SUE problem.

### 3 AN ALGORITHM FOR THE CNL-SUE PROBLEM

Damberg et al. (1996) extended the DSD algorithm of Larsson and Patriksson (1992) to solve the MNL-SUE problem. This path-based method iteratively solves sub-problems that are generated through partial linearization of the objective function. The new iteration solution is found as a convex combination of the solution of the linearized sub-problem and the previous iteration solution. This section presents the adaptation of the method to the CNL-SUE problem.

Suppose that at iteration  $n$  a feasible path-flow solution is given. The term  $Z_l$  in formulation (2) is linearized, which amounts to assuming that travel costs are fixed at their current values.

The resulting objective function of the sub-problem is given by:

$$\text{Min } \bar{Z} = \sum_{rs} \sum_k c_k^{rs(n)} f_k^{rs} + \frac{\mu}{\theta} \sum_{rs} \sum_m \sum_k f_{mk}^{rs} \ln \frac{f_{mk}^{rs}}{(\alpha_{mk}^{rs})^{1/\mu}} + \frac{1-\mu}{\theta} \sum_{rs} \sum_m \left( \sum_k f_{mk}^{rs} \right) \ln \left( \sum_k f_{mk}^{rs} \right) \quad (3)$$

$c_k^{rs(n)}$  is the travel cost on path  $k$  based on the vector of path flows at iteration  $n$ . The solution to this sub-problem is given by the CNL model route choices as follows:

$$h_k^{rs(n)} = q^{rs} \frac{\exp \left[ -\theta c_k^{rs(n)} + \ln \sum_m \alpha_{mk}^{rs} \left( \sum_l \alpha_{ml}^{rs} \exp(-\theta c_l^{rs(n)}) \right)^{\mu-1} \right]}{\sum_j \exp \left[ -\theta c_j^{rs(n)} + \ln \sum_m \alpha_{mj}^{rs} \left( \sum_l \alpha_{ml}^{rs} \exp(-\theta c_l^{rs(n)}) \right)^{\mu-1} \right]} \quad (4)$$

Following Larsson et al. (1993), if the vector  $h^{(n)} - f^{(n)}$  is nonzero, it defines a descent direction with respect to  $Z$ .

The new solution is given by:

$$f_k^{rs(n+1)} = f_k^{rs(n)} + \lambda^{(n)} \left( h_k^{rs(n)} - f_k^{rs(n)} \right) \quad \forall k, \forall rs \quad (5)$$

$\lambda^{(n)}$  is the step size in iteration  $n$ . In this paper we consider three different methods to calculate the step size: an exact calculation of the optimal step size, approximation of the optimal step size using Armijo's rule and application of pre-determined step sizes, as in the MSA algorithm.

An exact optimal step size is calculated using the golden section line search method as the optimal solution of the following problem:

$$\lambda^{(n)} = \arg \min_{\lambda \in [0,1]} Z \left[ f^{(n)} + \lambda \left( h^{(n)} - f^{(n)} \right) \right] \quad (6)$$

The exact line search may be computationally expensive to perform in the case of the CNL-SUE problem. This is because the variable of interest is  $f_{mk}^{rs}$ . The dimension of this variable may be very large even for moderately sized networks. Consequently the number of operations required to calculate the objective function value and the overall effort to find the

optimal step size may be very large. An alternative approach is to use Armijo's approximate step size rule (Armijo, 1966), which is defined by:

$$\lambda^{(n)} = \beta^{m_k} \quad (7)$$

$m_k$  is the first integer,  $m \geq 0$ , which satisfies:

$$Z(f^{(n)}) - Z(f^{(n)} + \beta^m (h^{(n)} - f^{(n)})) \geq -\varepsilon \beta^m \nabla Z(f^{(n)}) (h^{(n)} - f^{(n)}) \quad (8)$$

$0 < \beta < 1$  and  $0 < \varepsilon < 1$  are parameters.

The Armijo rule can be used if the objective function is Lipschitz continuous. Assuming a monotonic increasing cost function, the objective function (2) is continuous and convex. For finite demand, its derivatives exist and are bounded, and therefore it is also Lipschitz continuous.

Finally, the simplest approach to the step size calculation is the use of pre-determined step sizes. These approaches require very little effort in each iteration, but may require many more iterations to reach convergence. We applied the following step size rule, which simplifies the proposed algorithm to a path-based MSA method:

$$\lambda^{(n)} = \frac{1}{n+1} \quad (9)$$

The flows calculated in equation (5) are then used to update link costs and path costs. A new sub-problem with the updated path costs is solved using equation (4), to produce a new descent direction. This iterative process continues until the convergence criterion is satisfied.

Larsson et al. (1993) show that since  $Z$  is strictly convex, the sequence  $\{f^{(n)}\}$  of path flows converges to the unique solution of the problem from any feasible initial flow  $f^{(0)}$ .

#### **4 CASE STUDY: THE WINNIPEG NETWORK**

The Winnipeg network database provided in the EMME/2 software (INRO, 1999) is used for testing the CNL-SUE algorithm. The network is composed of 948 nodes (154 of which are centroids), 2,535 links and 4,345 OD pairs with positive demand. The total demand on the network is for 54,459 trips. The volume-delay function for each link is based on the BPR formula with link-specific parameters, calculated from the original EMME/2 data.

Routes were generated prior to the assignment, using a combination of the link elimination method of Azevedo et al. (1993) and the penalty method of De La Barra et al. (1993), with a penalty of 5% increase travel time on the shortest path links. Only acyclic paths were considered in these methods. A total of 174,491 unique routes were generated for all OD pairs (average of 40.1 routes per OD pair). The maximum possible number of routes generated for each OD pair was 50. Inspection on the routes generated for the different OD pairs reveals that the choice set used for the analysis includes both completely disjoint routes and very similar routes. This was expected due to the methods (link penalty and link elimination) selected to generate the routes: the link elimination method produces disjoint routes (because

of the removal of all links belonging to the shortest path) and the link penalty method produces similar routes because of the low penalty (5% increase link travel time) used to find the subsequent routes.

The results are presented in two subsections. The first focuses on the computational performance of the algorithm. The second compares results of the CNL-SUE assignment to those of the MNL-SUE model.

#### 4.1 Computational performance

The three algorithm variations, which differ in the step size calculation methods, were run on a PC with a Pentium-IV 3.0 GHz CPU and 512 MB RAM. In all cases, the termination criterion for the algorithms was based on the internal inconsistency of the solution:

$$RMSE^{(n)} = \sqrt{\frac{1}{K} \sum_{rs} \sum_k (h_k^{rs(n)} - f_k^{rs(n)})^2} \quad (10)$$

$K$  is the number of routes in the choice sets. The convergence criteria used in these tests is

$$RMSE^{(n)} \leq 0.001.$$

Figure 1 shows an example of the CPU time as a function of the number of iterations (for  $\mu=0.75$ ,  $\theta=0.25$ ). With all three methods the CPU time is linear with the number of iterations.

As expected, the CPU time per iteration is lowest with the MSA algorithm (80 seconds) and highest (2270 seconds) when the step size is calculated with the golden section method.

However, the MSA algorithm requires a very large number of iterations to converge compared to the other methods, and consequently it is slower overall. The Armijo rule performs well compared to the other two methods. Similar to the golden section method, it requires a relatively small number of iterations to converge, but the amount of work per iteration is low (384 seconds per iteration), since the objective function is calculated relatively few times. Bekhor et al. (2007) present further sensitivity tests showing the difference in performance between the three step size calculation methods and supporting the superiority of the Armijo rule over the other two methods. Therefore, in all subsequent tests performed in this paper, the step size calculation is based on the Armijo rule.

[Insert Figure 1 here]

Figure 2 shows the effect of the values of the parameters  $\theta$  and  $\mu$  on the CPU times for the CNL-SUE model. Note that when the parameter  $\mu$  is equal to 1, the CNL-SUE collapses to the MNL-SUE model. The computational effort increases with increasing values of the dispersion parameter  $\theta$ . This is expected since with larger values the auxiliary solution tends to allocate more flow to the shortest path compared to other routes. The outcome of this process is smaller step sizes to maintain feasibility, slowing the overall convergence.

[Insert Figure 2 here]

The computational effort also increases when the nesting parameter  $\mu$  decreases, i.e., when the correlations among overlapping routes are higher. Higher correlations indirectly introduce impacts of flows on one route on other routes, and so may slow convergence. This result is discussed in detail in Prashker and Bekhor (2000).

The MNL-SUE formulation can be solved faster in almost all cases. The difference in CPU times increases with the value of the dispersion parameter  $\theta$ . However, even in the most extreme cases, the CNL-SUE CPU time is longer by a factor that is less than 3.5, and much smaller in most cases.

The results presented in Figure 2 referred to the computational effort required to reach convergence for a given RMSE equal to 0.001. Figure 3 shows the progress of the algorithm until it reaches convergence by plotting the accuracy of the solution, as measured by the RMSE statistic, as a function of the CPU time. Note that the Y axes in the graphs are in logarithmic scales. The graphs again show that the effort to reach a given RMSE level increases with  $\theta$  and decreases with  $\mu$ . These effects are more pronounced when a more accurate solution is required (i.e. small RMSE value). An interaction effect also exists – the effect of  $\mu$  is more pronounced for larger  $\theta$  values, and the effect of  $\theta$  is more pronounced for smaller  $\mu$  values.

[Insert Figure 3 here]

The Winnipeg network is mildly congested. In order to investigate the effect of the level of congestion on the computational effort we scaled the demand for travel in the network by a constant factor and calculated the CNL-SUE assignment with the modified demand. Figure 4 shows the effect of the demand scaling factor on the computational performance for different values of the parameters  $\theta$  and  $\mu$ . The graphs show the CPU times needed to reach convergence (RMSE=0.001). Y-axes are in logarithmic scale, and the X-axes represent the demand scaling factor. As expected, the CPU times increase for increasing demand values, a result well-known in the literature for deterministic user equilibrium models. However, it is noticeable that also when the network is quite congested (1.4 times the original demand matrix), the CPU times are not longer than 10,000 seconds to reach convergence.

[Insert Figure 4 here]

The graphs above also present results for the MNL-SUE model for comparison. As expected, the simpler MNL model results in faster convergence. However, the CNL-SUE model performs quite well, given the additional computational effort required to calculate the route choice probabilities.



## 4.2 Comparison between MNL-SUE and CNL-SUE results

This section compares link flows between the MNL-SUE and CNL-SUE assignment models for the Winnipeg network. The two models were compared on simple networks with constant travel times by Prashker and Bekhor (1998).

### 4.2.1 Influence of Network Parameters

In the case that the choice between routes is based on travel times only, the dispersion parameter  $\theta$  can be interpreted as the travel time coefficient of the route choice model. In the comparison identical values of this parameter were used with the MNL and CNL models.

In addition to the dispersion parameter, the CNL model contains additional parameters to account for the similarity among routes. The allocation parameters  $\alpha_{mk}$  can be calculated directly from the network topology (Prashker and Bekhor 1998). The nesting coefficient  $\mu$  can be interpreted as the “weight” given to the similarity among routes: when  $\mu=1$ , the CNL model collapses to the MNL model, regardless of the values of  $\alpha_{mk}$ , and when  $\mu$  tends to zero, the resulting effect is a deterministic choice probability.

The discussion above is valid when the travel times are constant. However, in the case of MNL-SUE and CNL-SUE models travel times depend on traffic flows, and so the combined effect of similarity and congestion needs to be evaluated. Figure 5 and Figure 6 show the

effect of the values of the parameters  $\theta$  and  $\mu$  and of the demand for travel on the difference between CNL-SUE and MNL-SUE results, respectively. The deviation between the two models in Figure 5 is measured by the root mean squared error of path flows:

$$RMSE = \sqrt{\frac{1}{K} \sum_{rs} \sum_k (f_k^{rs(CNL)} - f_k^{rs(MNL)})^2} \quad (11)$$

$K$  is the total number of routes and  $K_{rs}$  is the number of routes for the specific origin-destination pair  $rs$ .

In figure 6, the deviation is measured by the root mean square percentage error:

$$RMSPE_1 = \sqrt{\frac{1}{K} \sum_{rs} \sum_k \left( \frac{f_k^{rs(CNL)} - f_k^{rs(MNL)}}{(f_k^{rs(CNL)} + f_k^{rs(MNL)})/2} \right)^2} \quad (12)$$

[Insert Figure 5 here]

[Insert Figure 6 here]

The results presented in Figure 5 are consistent with the model theory in that the difference between the models decreases for increasing values of  $\mu$ . Figure 6 shows an interesting result that the difference between CNL-SUE and MNL-SUE models is practically constant for different demand levels. It only slightly increases for the highest level of demand. The conclusion from these figures is that the overall difference between the models can be very significant depending on the network topology and network parameters.

#### 4.2.2 Link Flow Patterns

The results of the previous section evaluated the overall difference between the models. In this section the link flow patterns are compared in further detail. The comparison is based on fixed values for  $\theta$  and  $\mu$ .  $\theta$  was set to 0.5, which indicates that given a 5-minute difference between two paths, about 8% of the drivers will chose the route with the higher cost. For the CNL model  $\mu=0.5$  was assumed. This means that if there are no congestion effects, the CNL route choice probability will allocate less flow for routes that contain overlapping links, compared to the MNL route choice probability model. Note that in a hypothetical case that all paths are disjointed (i.e. there are no links in common), the CNL and MNL route choice probabilities will produce the same result, regardless of the value of  $\mu$ .

Figure 7 illustrates the link flow difference between MNL-SUE and CNL-SUE results. Links coded in darker color indicate that MNL-SUE link flows are higher than CNL-SUE link flows. Note the concentration of these links around the city center. This is explained by the higher number of links in the city center, and consequently higher similarity among routes passing through the center. The CNL-SUE model allocates less flow in these routes compared to MNL-SUE because it considers the similarity among routes.

[Insert Figure 7 here]

### 4.2.3 Influence of Route Set on Link Flows

This section presents a comparison between MNL-SUE and CNL-SUE equilibrium route flows and link flows for different choice set sizes. CNL-SUE and MNL-SUE assignments were conducted with varying maximum number of routes generated for each OD pair: starting at only two routes and gradually increasing it up to 50 routes for each OD pair. In addition to equations (11) and (12), the following goodness-of-fit measure was used to quantify the deviations between the two models:

$$RMSPE_2 = \sqrt{\frac{1}{K} \sum_{rs} \sum_k \left( \frac{f_k^{rs(CNL)} - f_k^{rs(MNL)}}{q_{rs} / K_{rs}} \right)^2} \quad (13)$$

$K$  is the total number of routes and  $K_{rs}$  is the number of routes for the specific origin-destination pair  $rs$ . Equations (11) and (12) are used both at the route and link levels, and equation (13) is used for route comparison only. The rationale for using equation (13) is to avoid large contributions to the error by routes which carry a very small fraction of the demand, which may happen when the number of routes in the choice set increases.

Figure 8 contains two graphs: the left figure presents percent differences (RMSPE), and the right figure presents absolute differences (RMSE). The relative difference between CNL-SUE and MNL-SUE is smaller in terms of link flows compared to path flows. This result is expected due to aggregation of path flows into link flows. Apart from the RMSE for path

flows, all other measurements increase with increasing choice set size. Since the fixed routes generated are based on variations of the shortest path route, there is a certain degree of similarity among these routes (because of the common links). Therefore, as the number of routes in the choice set increase, the similarity among routes also increases. The CNL model accounts for similarity, and consequently there is an increase of the relative difference between CNL and MNL path flows for increasing set sizes.

An interesting result is found with respect to the RMSE for link flows. When the number of routes increases, this statistic also increases. This means that MNL-SUE and CNL-SUE link flows can be quite different depending on the number of routes used in the assignment. Recall that from a purely mathematical point of view the "true" equilibrium solution is achieved if all routes are included in the choice set. Since we confine the number of routes to a finite (and small) number, the equilibrium flows may be quite different than the "true" flow pattern. However, it should also be noted that the mathematically "true" solution may be unreasonable from a behavioral perspective.

[Insert Figure 8 here]

## **5 SUMMARY AND CONCLUSIONS**

This paper discussed path-based algorithms to solve the CNL-SUE problem. The motivation of the paper was to incorporate a more realistic route choice model into the traffic assignment problem. The results of the paper showed that it is possible to implement such algorithms at affordable computer resources. In particular, the Armijo step size rule was found to perform well for the network tested, but more experiments need to be conducted to verify if the rule works well also in other cases.

The CNL route choice model can overcome deficiencies of the MNL model. Consequently, the CNL-SUE model can better represent traffic phenomena compared to the MNL-SUE model. This is also true for other equilibrium procedures such as MNP-SUE and PCL-SUE, as well as mixed logit models. Further research will address the trade-offs, specifically, algorithm performance versus path-flow difference, between these models and the CNL-SUE model.

The tests presented in this paper focused on the effect of several parameters and inputs: the dispersion parameter, the nesting coefficient, the level of demand and the size of the route set.

The results demonstrate that differences between CNL-SUE and MNL-SUE can be quite pronounced.

In our comparisons, route choices were assumed to depend only on travel times. The problem formulation is flexible in the definition of route utilities and can accommodate additional explanatory variables, such as travel cost, within a generalized cost function. However, more complex route utility functions have not yet been implemented in the context of traffic assignment models. Furthermore, the results presented in this paper are based on the common assumptions of the standard user equilibrium model: static assignment, fixed demand, separable volume-delay function and a single user class. Additional research is needed to extend and verify the CNL-SUE model for more general problems.

## REFERENCES

- Akamatsu, T. (1996), Cyclic flows, Markov process and stochastic traffic assignment, *Transportation Research Part B*, **30**(5), 369-386.
- Armijo, L. (1966), Minimization of Functions having Lipschitz Continuous First Derivatives, *Pacific Journal of Mathematics*, **16**, 1–3.
- Azevedo, J.A., M.E.O. Santos Costa, J.J.E.R. Silvestre Madera, and E.Q. Vieira Martins (1993), An Algorithm for the Ranking of Shortest Paths, *European Journal of Operational Research*, **69**, 97-106.

- Bekhor, S., Ben-Akiva, M.E., and S. Ramming (2002), Adaptation of Logit Kernel to Route Choice Situation, *Transportation Research Record*, **1805**, 78-85.
- Bekhor, S., and J.N. Prashker (1999), Formulations of Extended Logit Stochastic User Equilibrium Assignments, In A. Ceder (ed.), *Proceedings of the 14th International Symposium on Transportation and Traffic Theory*, Pergamon Press, 351-372.
- Bekhor, S. and J.N. Prashker (2001), Stochastic user equilibrium formulation for generalized nested logit model, *Transportation Research Record*, **1752**, 84-90.
- Bekhor, S., Reznikova L., Toledo T. (2007), Application of the Cross Nested Logit Route Choice Model in Stochastic User Equilibrium Traffic Assignment, 88<sup>th</sup> Annual Meeting of the Transportation Research Board, January 2007, Washington, D.C.
- Bell, M.G.H., 1995, Alternatives to Dial's Logit Assignment Algorithm, *Transportation Research Part B*, **29**(4), 287-295.
- Ben-Akiva, M., and M. Bierlaire (1999), Discrete Choice Methods and Their Applications to Short Term Travel Decisions, In Randolph W. Hall (ed.) *Handbook of Transportation Science*,
- Cascetta, E., A. Nuzzolo, F. Russo, and A. Vitetta (1996), A Modified Logit Route Choice Model Overcoming Path Overlapping Problems: Specification and Some Calibration



- Results for Interurban Networks, In J.B. Lesort (ed.) *Proceedings of the International Symposium on Transportation and Traffic Theory*, Lyon, 697-711.
- Chen A., P. Kasikitwiwat, and Z. Ji (2003), Solving the Overlapping Problem in Route Choice with Paired Combinatorial Logit Model, *Transportation Research Record*, **1857**, 65 – 73.
- Daganzo, C.F., and Sheffi, Y. (1977), On Stochastic Models of Traffic Assignment, *Transportation Science*, **11**, 253-274.
- Damberg, O., Lundgren, J.T., and Patriksson, M. (1996), An Algorithm for the Stochastic User Equilibrium Problem, *Transportation Research Part B*, **30**, 115-131.
- De La Barra, T., B. Perez, and J. Anez (1993), Multidimensional Path Search and Assignment, *Proceedings of the 21st PTRC Summer Annual Meeting*, Manchester, England, 307-319.
- Dial, R.B. (2001), Equilibrium Logit Traffic Assignment: Elementary Theory and Algorithm. Paper presented at the 80th Transportation Research Board Annual Meeting, Washington DC, 2001.
- Fisk, C. (1980), Some Developments in Equilibrium Traffic Assignment, *Transportation Research Part B*, **14**, 243-255.

Gliebe, J.P., Koppleman, F.S., and A. Ziliaskopoulos (1999), Route Choice Using a Paired Combinatorial Logit Model, Preprints of the 78th Annual Meeting of the Transportation Research Board, Washington, D.C.

INRO Consultants (1999), *Emme/2 User's Manual: Release 9.2*. Montréal.

Larsson, T. and M. Patriksson (1992), Simplicial Decomposition with Disaggregated Representation for the Traffic Assignment Problem, *Transportation Science*, **26**(1), 4-17.

Larsson, T., Migdalas A., and M. Patriksson (1993), A Partial Linearization Algorithm for the Traffic Assignment Problem, *Optimization*, **28**, 47-61.

Maher, M. (1998), Algorithms for Logit-based Stochastic User Equilibrium Assignment, *Transportation Research Part B*, **32**(8), 539-549.

Prashker, J.N., and S. Bekhor (1998), Investigation of Stochastic Network Loading Procedures, *Transportation Research Record*, **1645**, 94-102.

Prashker, J.N., and S. Bekhor (2000), Congestion, Stochastic, and Similarity Effects in Stochastic User-Equilibrium Models, *Transportation Research Record*, **1733**, 80-87.

Ramming, M.S. (2001), Network Knowledge and Route Choice. Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge.

Sheffi, Y., and W.B. Powell (1982), An Algorithm for the Equilibrium Assignment Problem with Random Link Times. *Networks*, **12**, 191-207.

Vovsha, P. and S. Bekhor (1998), The Link Nested Logit Model: Overcoming the Route Overlapping Problem, *Transportation Research Record*, **1645**, 133-142.

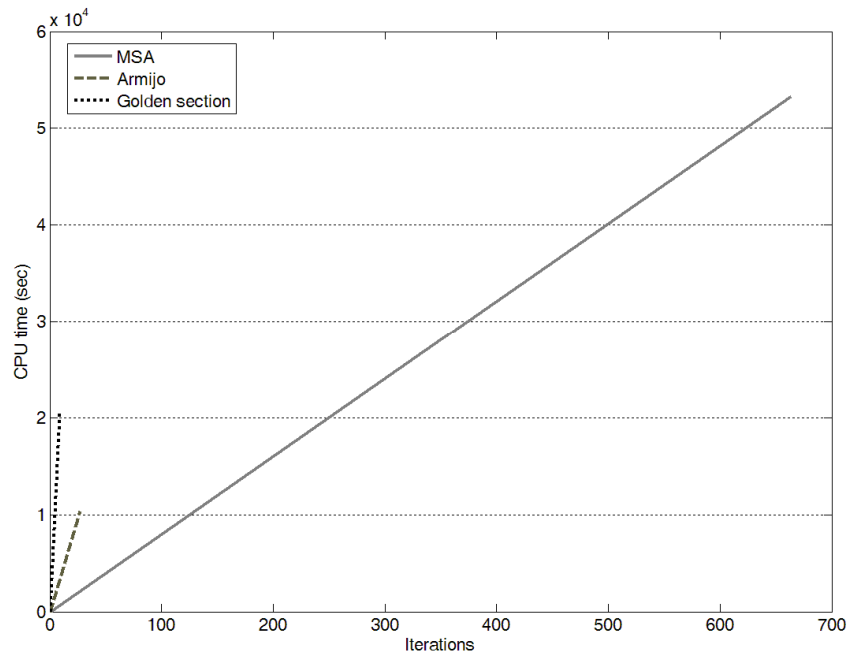


Figure 1. CPU times for the three step size calculation methods.

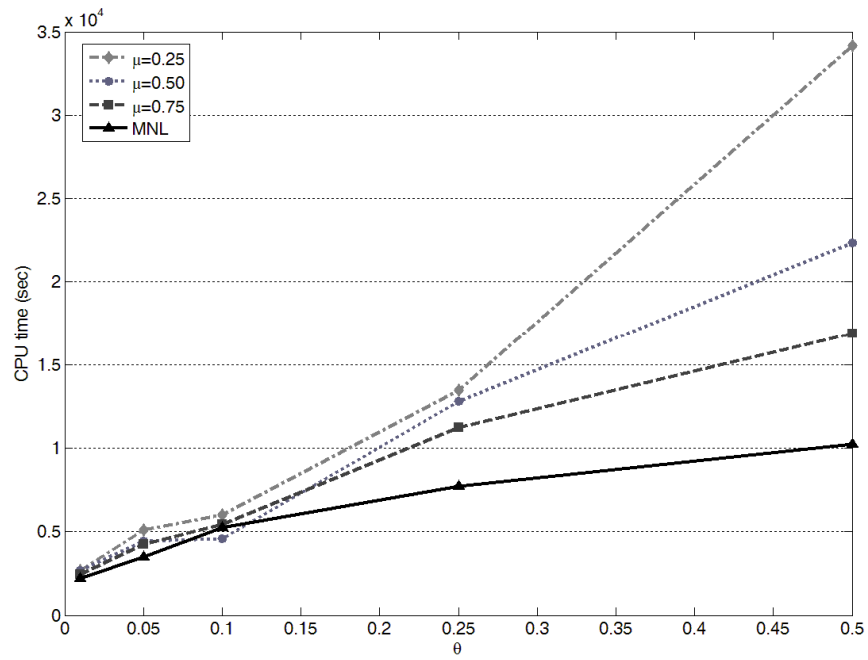


Figure 2. Effect of the values of the parameters  $\theta$  and  $\mu$  on the CPU time.

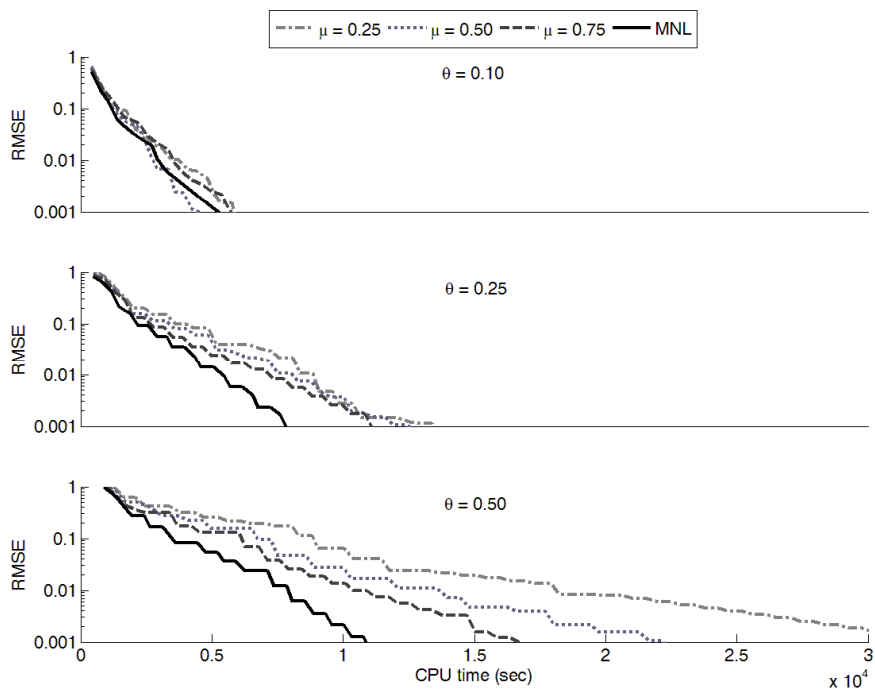


Figure 3. Accuracy of the solution as a function of the CPU time.

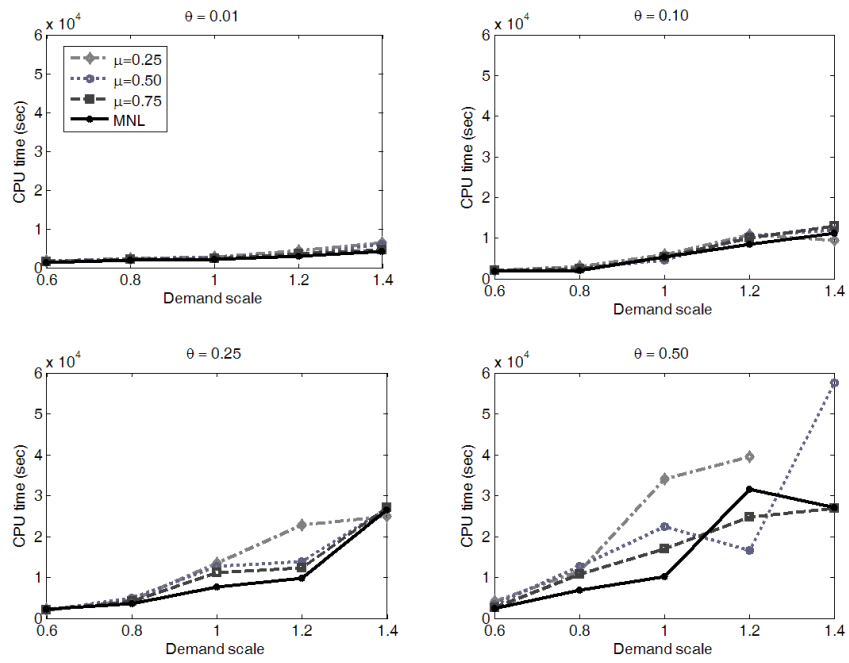


Figure 4. Effect of the travel demand on the CPU Time.

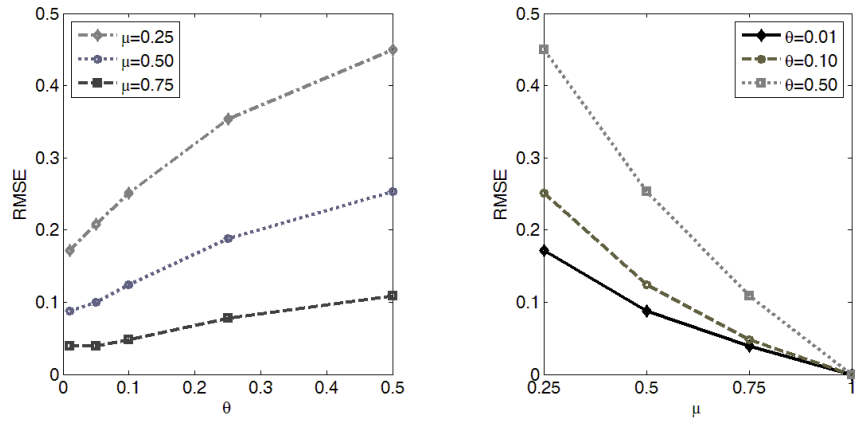


Figure 5. Effect of the values of the parameters  $\theta$  and  $\mu$  on the difference between CNL-SUE and MNL-SUE results.



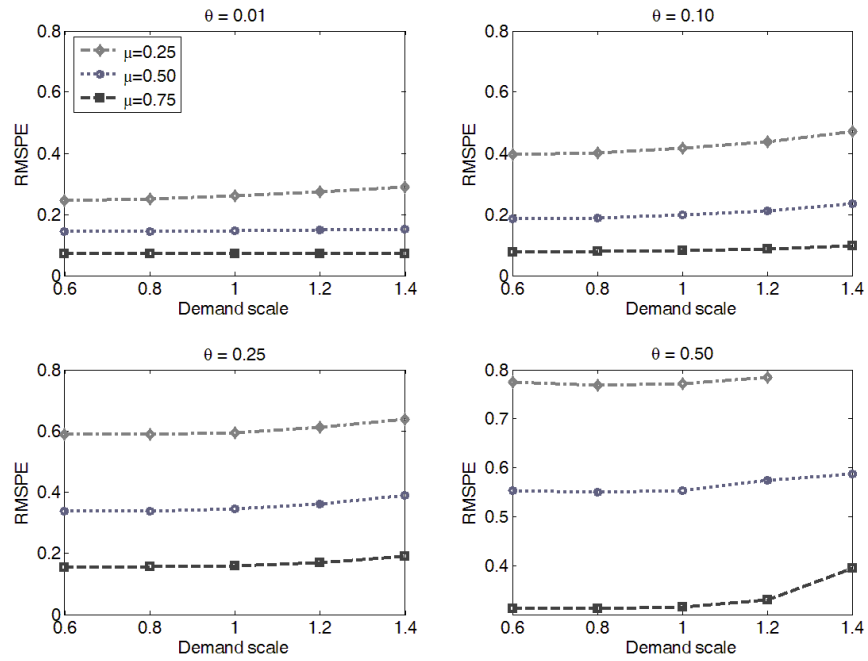


Figure 6. Effect of the demand for travel on the difference between CNL-SUE and MNL-SUE results.



Figure 7 Difference in link flows between MNL-SUE and CNL-SUE models.

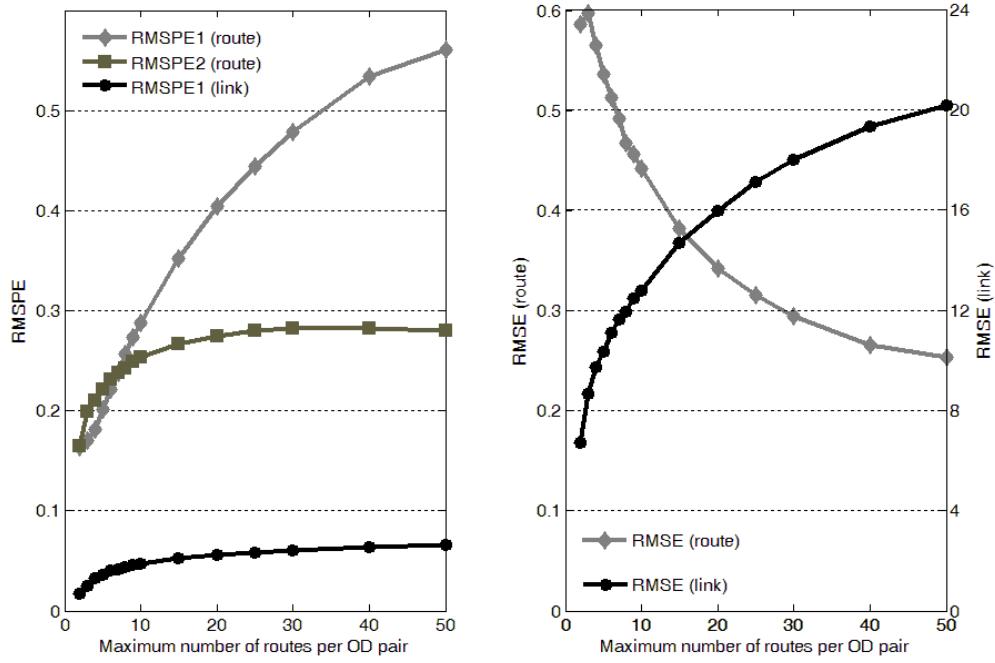


Figure 8 Effect of the route set size on the difference between CNL-SUE and MNL-SUE results.